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A CONSTRUCTION OF NEW FAMILIES OF MINIMAL LAGRANGIAN SUBMANIFOLDS VIA TORUS ACTIONS

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Abstract

In this paper we investigate connections between minimal Lagrangian submanifolds and holomorphic vector fields in Kähler manifolds. Our main result is: Let M^{2n} $(n \geq 2)$ be a Kähler-Einstein manifold with positive scalar curvature with an effective, structure-preserving action by an n-torus T^n. Then precisely one regular orbit L of the T^n-action is a minimal Lagrangian submanifold of M. Moreover there is an $(n-1)$-torus $T^{n-1} \subset T^n$ and a sequence of non-flat immersed minimal Lagrangian tori L_k in M such that all L_k are invariant under T^{n-1} and L_k locally converge to L (in particular the supremum of the sectional curvatures of L_k and the distance between L and L_k go to 0 as $k \mapsto \infty$). This result is new even for $M = \mathbb{C}P^n$ for $n \geq 3$.

1. Introduction

This paper constitutes a part of author's program in trying to understand the geometry of minimal Lagrangian submanifolds in Kähler-Einstein manifolds. Kähler-Einstein manifolds are just Kähler manifolds whose Ricci form is proportional to their Kähler form. The existence theory of Kähler-Einstein metrics on complex manifolds was one of the major recent achievements in complex differential geometry. One should mention the work of Aubin [2] and Yau [18] for the case of negative scalar curvature, Yau's resolution of the Calabi conjecture and the existence of Calabi-Yau metrics [18] and the work of Tian [17] on the existence of Kähler-Einstein metrics with positive scalar curvature.

On a given Kähler-Einstein manifold M one has a class of minimal Lagrangian submanifolds of M. Those are just Lagrangian submanifolds

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of M, which in addition are minimal submanifolds of M, i.e., their mean curvature vector field vanishes. Compact minimal Lagrangian submanifolds of M are just the critical points of a natural variational problem, namely minimal Lagrangian submanifolds of M are the critical points for the volume functional on the space of Lagrangian submanifolds in M (see [15]).

The geometry of minimal Lagrangian submanifolds in a given Kähler-Einstein manifold M and the structure of their moduli-space depends strongly on the sign of the scalar curvature s of M. If the scalar curvature s of M is zero then M is a Calabi-Yau manifold and minimal Lagrangian submanifolds of M are just the Special Lagrangian submanifolds of M (see [16]). The Special Lagrangian submanifolds are calibrated and they come in smooth, finite-dimensional families. Special Lagrangian submanifolds have recently attracted a lot of attention due to their role in the Strominger-Yau-Zaslow mirror symmetry conjecture [16].

If the scalar curvature s of M is negative then minimal Lagrangian submanifolds of M have interesting strict volume minimization properties which we hope to explore in a forthcoming paper. In this paper we will mostly be interested in the case when the scalar curvature of M is positive. Our aim is to understand certain global properties that minimal Lagrangian submanifolds possess and to construct some new examples of minimal Lagrangian submanifolds. The main tools in our investigation will be holomorphic vector fields in the vicinity of minimal Lagrangian submanifolds on M (Section 2) and holomorphic isometries of M (Sections 3 and 4). Our main result is a construction of new families of minimal Lagrangian submanifolds in toric Kähler-Einstein manifolds (Theorem 4.3.1 in Section 4). The paper is organized as follows:

In Section 2 we study some global connections between minimal Lagrangian submanifolds and holomorphic vector fields in their vicinity in a Kähler manifold M. In Section 2.1 we present some basic properties of minimal Lagrangian submanifolds and holomorphic vector fields used throughout the paper. In Section 2.2 we prove the following Theorem:

Theorem 1. *Let N be a Kähler manifold, L be a compact oriented immersed minimal Lagrangian submanifold of N without boundary and V be a holomorphic vector field defined in a neighbourhood of L in N. Then if* $\mathrm{div}(V)$ *is the holomorphic divergence of V, we have*

$$\int_L \mathrm{div}(V) = 0.$$

We also give a simple application of this theorem to minimal Lagrangian submanifolds in $\mathbb{C}P^n$ (see Corollary 2.2.1). While this theorem is not directly relevant to the main result of the paper, we nevertheless include it for its own interest.

In Sections 3 and 4 we use structure-preserving torus actions on Kähler-Einstein manifolds with positive scalar curvature to construct invariant minimal Lagrangian submanifolds. Our strategy for doing this is as follows:

We begin with Section 3.1 reviewing the geometry of the total space of the canonical bundle $K(M)$ of a Kähler-Einstein manifold M with positive scalar curvature. We will see that $K(M)$ is naturally a Calabi-Yau manifold. Moreover there is a correspondence between minimal Lagrangian submanifolds of M and Special Lagrangian submanifolds of $K(M)$, invariant under a certain radial vector field Y on $K(M)$ (see Lemmas 3.1.1 and 3.1.2 of Section 3.1). Thus we recast the problem of constructing minimal Lagrangian submanifolds of M to a problem of constructing Special Lagrangian submanifolds of $K(M)$, invariant under the Y-flow.

In Section 3.2 we suppose that we have a structure-preserving action by a k-dimensional torus T^k on M. We will be interested in finding the T-invariant minimal Lagrangian submanifolds of M. Now the T^k-action on M induces a T^k-action on $K(M)$. We will see that there are canonical moment maps μ on M and μ' on $K(M)$ for these actions. Let $Z \subset M$ be the zero set of μ and $\pi : K(M) \to M$ be the projection. Then the zero set of μ' on $K(M)$ is $Z' = \pi^{-1}(Z)$. Suppose that T acts freely on $Z'' = Z' - Z$ (here we view $Z \subset M$ and M is the zero section of $K(M)$). Then we have a symplectic reduction $Q = Z''/T$. We will see that Q has a natural holomorphic volume form φ' and a Kähler metric ω' and Special Lagrangian submanifolds of (Q, ω', φ') lift to T-invariant Special Lagrangian (SLag) submanifolds of $K(M)$. Also the vector field Y is tangent to Z'' and it projects to a vector field Y' on Q. Thus we reduced the problem of finding T-invariant minimal Lagrangian submanifolds of M to a problem of finding SLag submanifolds of Q, invariant under the flow of Y'.

In Section 3.3 we assume that $k = n-1$. In that case Q had complex dimension 2. Let $X \subset Z''$ be the set of elements of Z'' of unit length in $K(M)$ and $S = X/T \subset Q$. Then S is a compact, 3-dimensional submanifold of Q. We will see that there is a non-vanishing vector field W on S such that there is a correspondence between Y'-invariant SLag submanifolds of Q and the trajectories of the W-flow on S.

Next we would like to develop a criterion to see that T^{n-1} acts freely on Z''. We also would like to understand the periodic orbits of the vector field W on S (to construct immersed minimal Lagrangian tori on M). We can do it if we assume that M is a toric K-E manifold (see Section 4). In this case we can prove the following theorem, which is the main theorem of this paper:

Theorem 2. *Let M^{2n} $(n \geq 2)$ be a Kähler-Einstein manifold with positive scalar curvature with an effective structure-preserving T^n-action. Then precisely one regular orbit L of the T^n-action is a minimal Lagrangian submanifold of M. Moreover there is an $(n-1)$-torus $T^{n-1} \subset T^n$ and a sequence of non-flat immersed T^{n-1}-invariant minimal Lagrangian tori $L_k \subset M$ such that L_k locally converge to L (in particular the supremum of the sectional curvatures of L_k and the distance between L and L_k go to 0 as $k \to \infty$).*

Here by local convergence we mean the following: The distance between L_k and L goes to 0 as $k \to \infty$. Also for any point $l \in L$ we can choose a neighbourhood U_l of l in M such that for k large enough $L_k \bigcap U_l$ is a finite union $\bigcup L_k^j$ of submanifolds of the form $L_k^j = \exp(v_k^j)(L \bigcap U_l)$, where v_k^j is a normal vector field to L on $L \bigcap U_l$. Moreover any subsequence v_k^j converges to 0 in the C^∞ topology as $k \to \infty$.

This result is new even for $M = \mathbb{C}P^n$ for $n \geq 3$. For $n = 2$ examples of non-flat S^1-invariant immersed minimal Lagrangian tori in $\mathbb{C}P^2$ were constructed in [5] and in [10] using harmonic maps. After our paper was completed D. Joyce published a preprint [11], in which he in particular constructs Special Lagrangian cones in $\mathbb{C}^{n+1}$, invariant under a linear action of T^{n-1}. Those cones project to non-flat immersed minimal Lagrangian tori in $\mathbb{C}P^n$, invariant under T^{n-1}.

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2. Global connections between holomorphic vector fields and minimal Lagrangian submanifolds

Let (M^{2n}, ω) be a Kähler manifold. In this section we will establish a global relation between minimal Lagrangian submanifolds of M and holomorphic vector fields in their vicinity on M (Theorem 2.2.1 of Section 2.2). We begin however with Section 2.1, presenting the basic facts needed throughout the paper:

2.1 Basic properties

Let (M^{2n}, ω) be a Kähler manifold. In this section we will discuss holomorphic vector fields on M and present some basic facts on minimal Lagrangian submanifolds of M. The results of this section are essentially known, though they are often stated in different terms in the literature.

First we discuss holomorphic vector fields on M. Let V be a vector field defined on some open subset U of M. The following proposition is elementary and well known (see [12], Proposition 4.1):

Proposition 2.1.1. *The following conditions are equivalent:*

1) *The flow of V commutes with the complex structure J on M.*

2) *For any point $m \in U$ the endomorphism $X \mapsto \nabla_X V$ of $T_m M$ is J-linear on $T_m M$.*

3) *The vector field $V - iJV$ gives a holomorphic section of $T^{(1,0)}U$.*

A vector field V satisfying the conditions of Proposition 2.1.1 is called a *holomorphic* vector field. Let V be a holomorphic vector field on some open subset U of M and let m be a point in U. Since the endomorphism $X \to \nabla_X V$ is J-linear on $T_m M$ we can define

$$\mathrm{div}(V) = \mathrm{trace}_{\mathbb{C}}(X \to \nabla_X V). \tag{1}$$

Let $f = \mathrm{Re}\, f + i\mathrm{Im}\, f$ be a holomorphic function on U. From Condition 3) of Proposition 2.1.1 we deduce that the vector field $fV = \mathrm{Re}\, fV + \mathrm{Im}\, fJV$ is a holomorphic vector field on U. Moreover one easily computes that

$$\mathrm{div}(fV) = f\mathrm{div}(V) + V(f). \tag{2}$$

Let $K(M)$ be the canonical bundle of M, i.e., $K(M) = \Lambda^{(n,0)}T^*M$, the bundle of $(n, 0)$-forms on M. Let φ be a section of $K(M)$ over an open subset U of M (not necessarily a holomorphic section). Thus φ is an $(n, 0)$-form on U.

Proposition 2.1.2. *Let V be a holomorphic vector field on U. Then*

$$\mathrm{div}(V)\varphi = \mathcal{L}_V \varphi - \nabla_V \varphi.$$

Proof. Let $m \in U$. Pick a unitary basis $X_1, \dots, X_n$ of $T_m M$ (here $T_m M$ is viewed as Hermitian vector space with the complex structure J). Extend X_i to a unitary frame in a neighbourhood of m. Then

$$\begin{aligned} \mathcal{L}_V \varphi(X_1, \dots, X_n) = V(\varphi(X_1, \dots, X_n)) \\ - \Sigma \varphi(X_1, \dots, [V, X_n], \dots, X_n) \end{aligned} \tag{3}$$

and

$$\begin{aligned} \nabla_V \varphi(X_1, \dots, X_n) = V(\varphi(X_1, \dots, X_n)) \\ - \Sigma \varphi(X_1, \dots, \nabla_V X_i, \dots, X_n). \end{aligned} \tag{4}$$

Now $\nabla_V X_i = [V, X_i] + \nabla_{X_i} V$. We plug this into (4) and subtract (4) from (3) to deduce the statement of the proposition. q.e.d.

The following result is due to A. Futaki [6] and we include its proof for reader's convenience:

Lemma 2.1.1. *Let (M^{2n}, ω) be a Kähler-Einstein manifold with non-zero scalar curvature s and let $t = s/2n$. Let V be a holomorphic infinitesimal isometry defined on some neighbourhood U of M. Then the function $\mu = it^{-1}\mathrm{div}(V)$ is a moment map for the V-action on (M, ω).*

Proof. We need to prove that $d\mu = i_V \omega$. We shall prove it at a point m such that $V(m) \neq 0$. Pick an element φ of $K(M)$ over m which has unit length. Since the flow of V is given by holomorphic isometries we can extend φ to a unit length section of $K(M)$ invariant under the V-flow on some neighbourhood U of m. The section φ defines a connection 1-form ξ on U, given by $\xi \otimes \varphi = \nabla \varphi$. The Einstein condition tells that

$$id\xi = t\omega. \tag{5}$$

Since φ is V-invariant we deduce from Proposition 2.1.2 that

$$\mathrm{div}(V) = -\xi(V). \tag{6}$$

Also since φ is V-invariant and the flow of V is given by isometries, we deduce that ξ is also V-invariant. Thus

$$\begin{aligned} 0 = \mathcal{L}_V \xi &= d(\xi(V)) + i_V d\xi \\ &= -d(\mathrm{div}(V)) - iti_V \omega \quad \text{by} \quad (5) \text{ and } (6) \end{aligned}$$

and the lemma follows. q.e.d.

Next we discuss minimal Lagrangian submanifolds on M. Let L be an oriented Lagrangian submanifold of M. For any point $l \in L$ there is a unique element κ_l of $K(M)$ over l which restricts to the volume form on L (in the induced metric from M). Various k_l give rise to a section

$$\kappa_L : L \to K(M). \tag{7}$$

The section κ_L has constant length $\sqrt{2^n}$ over L and it defines a connection 1-form ξ on L for the connection on $K(M)$ over L by the condition $\xi \otimes \kappa_L = \nabla \kappa_L$. Here ∇ is the connection on $K(M)$, induced from the Levi-Civita connection on M. Since κ_L has constant length, ξ is an imaginary valued 1-form on L.

Let h be the trace of the second fundamental form of L (the mean curvature vector field of L). So h is a section of the normal bundle of L in M and we have a corresponding 1-form $\sigma = i_h\omega$ on L. The following fact is well-known, although it is often stated a bit differently in the literature (see [13] and [15]):

Lemma 2.1.2. $\sigma = i\xi$.

Proof. Let $l \in L$ and e be some vector in the tangent space to L at l. To compute $\xi(e)$ we need to compute $\nabla_e \kappa_L$. Take an orthonormal frame (v_j) of T_lL and extend it to an orthonormal frame in a neighbourhood U of l in L such that $\nabla^L v_i = 0$ at l (here ∇^L is the Levi-Civita connection of L). We get that

$$\begin{aligned}\nabla_e \kappa_L &= \kappa_L \cdot \nabla_e \kappa_L(v_1, \dots, v_n)\\ &= \kappa_L(e(\kappa_L(v_1, \dots, v_n)) - \Sigma \kappa_L(v_1, \dots, \nabla_e v_j, \dots, v_n)).\end{aligned}$$

Now $e(\kappa_L(v_1, \dots, v_n)) = 0$. Also clearly

$$\begin{aligned}\kappa_L(v_1, \dots, \nabla_e v_j, \dots, v_n) &= i\langle \nabla_e v_j, Jv_j\rangle\\ &= i\langle \nabla_{v_j} e, Jv_j\rangle = i\langle -e, J(\nabla_{v_j} v_j)\rangle.\end{aligned}$$

Here J is the complex structure on M. Thus we get that

$$\nabla_e \kappa_L = -i\langle e, Jh\rangle \kappa_L = -i\sigma(e)\kappa_l.$$

Here $h = \Sigma \nabla_{v_j} v_j$ is the trace of the second fundamental form of L. Thus $\sigma = i\xi$. q.e.d.

Thus L is *minimal* (i.e., $h = 0$) iff κ_L is parallel over L.

2.2 Integral of the divergence

We now can state and prove the main theorem of Section 2:

Theorem 2.2.1. *Let M be a Kähler manifold, L be a compact oriented immersed minimal Lagrangian submanifold of M without boundary and V be a holomorphic vector field defined in a neighbourhood of L in M. Then*

$$\int_L \operatorname{div}(V) = 0.$$

Proof. Let L be a minimal Lagrangian submanifold of M and V be a holomorphic vector field defined in a neighbourhood of L in M. Let κ_L be a section of $K(M)$ over L as in Equation (7). Since κ_L restricts to the volume form on L we have $\int_L \operatorname{div}(V) = \int_L \operatorname{div}(V)\kappa_L$. Let

$$\phi = i_V \kappa_L|_L. \tag{8}$$

ϕ is an $(n-1)$-form on L. We claim that

$$d\phi = \operatorname{div}(V)\kappa_L|_L. \tag{9}$$

Thus the assertion of the theorem will follow. To prove (9) let l be a point in L. By Lemma 2.1.2 we have that for any element w in the tangent bundle to L, $\nabla_w \kappa_L = 0$. We can extend κ_L to a section κ'_L of $K(M)$ over some neighbourhood Z of l in M such that for any element w in the normal bundle of L to M in $Z \bigcap L$ we'll have $\nabla_w \kappa'_L = 0$. Thus we'll have $\nabla \kappa'_L = 0$ at every point in $L \bigcap Z$ (here ∇ is the covariant derivative on M). This implies that $d\kappa'_L = 0$ at every point in $L \bigcap Z$ (here again the exterior derivative d is on M). Now we use Proposition 2.1.2 for V and $\varphi = \kappa'_L$. We deduce that

$$\operatorname{div}(V)\kappa'_L = \mathcal{L}_V \kappa'_L$$

at every point in $L \bigcap Z$. Also $\mathcal{L}_V \kappa'_L = d(i_V \kappa_L) + i_V(d\kappa_L)$ and $d\kappa'_L = 0$ along L. Thus we get

$$\operatorname{div}(V)\kappa_L = d\phi.$$

q.e.d.

Let us derive a simple corollary of Theorem 2.2.1:

Corollary 2.2.1. *Let L be a compact immersed oriented minimal Lagrangian submanifold of $\mathbb{C}P^n$ and let $(z_1, \dots, z_{n+1})$ be the homogeneous coordinates on $\mathbb{C}P^n$. Then we can't have $|z_1| > |z_2|$ at all points of L.*

Proof. Consider the following circle action on $\mathbb{C}P^n$:

$$e^{i\theta}(z_1, \dots, z_{n+1}) = (e^{i\theta} z_1, e^{-i\theta} z_2, z_3, \dots, z_{n+1}).$$

Let V be the vector field on $\mathbb{C}P^n$ generating this action. $\mathbb{C}P^n$ is Kähler-Einstein with scalar curvature 2n, hence by Lemma 2.1.1 the function $i\mathrm{div}(V)$ is a moment map for the S^1-action on $\mathbb{C}P^n$. We have computed in [7] that

$$i\mathrm{div}(V) = (|z_1|^2 - |z_2|^2)/\Sigma|z_i|^2.$$

In fact we can also deduce this from Theorem 2.2.1. Indeed the map $f = (|z_1|^2 - |z_2|^2)/\Sigma|z_i|^2$ is a moment map for the S^1-action on $\mathbb{C}P^n$, hence it differs from $i\mathrm{div}(V)$ by a constant c. Also the submanifold $L' = ((z_1, \dots, z_{n+1})||z_1| = |z_j|)$ is a minimal Lagrangian submanifold of $\mathbb{C}P^n$ (this is the Clifford torus, see [13]). Hence by Theorem 2.2.1 $\int_{L'} \mathrm{div}(V) = 0$. From this we deduce that $c = 0$, i.e., $i\mathrm{div}(V) = f$.

Let now L be an immersed oriented minimal Lagrangian submanifold of $\mathbb{C}P^n$. We have $\int_L \mathrm{div}(V) = 0$. Hence we obviously can't have $|z_1| > |z_2|$ everywhere on L. q.e.d.

3. Minimal Lagrangian submanifolds and the symplectic reduction of the canonical bundle

3.1 The Calabi construction and the connection between minimal and Special Lagrangian submanifolds

Let (M^{2n}, ω) be a Kähler-Einstein (K-E) manifold with positive scalar curvature. In this section we will show that the total space $K(M)$ of the canonical bundle of M has a natural Calabi-Yau structure. Moreover there is a correspondence between minimal Lagrangian submanifolds of M and certain Special Lagrangian submanifolds of $K(M)$. We begin this section by studying the geometry of $K(M)$.

Let $K(M)$ be the total space of the canonical bundle of M^{2n} and $\pi : K(M) \to M$ be the projection. There is a canonical $(n, 0)$-form

ρ on $K(M)$ defined by $\rho(a)(v_1,\dots,v_n) = a(\pi_*(v_1),\dots,\pi_*(v_n))$. Here $a \in K(M)$ and $v_1,\dots,v_n$ are tangent vectors to $K(M)$ at a. The form $\varphi = d\rho$ is a holomorphic volume form on $K(M)$. If $z_1,\dots,z_n$ are local coordinates on M then $dz_1 \wedge \dots \wedge dz_n$ is a local section of $K(M)$ over M which defines a coordinate function y on $K(M)$. The collections of holomorphic functions $(z_1,\dots,z_n,y)$ are coordinates on $K(M)$ and

$$\rho = y dz_1 \wedge \dots \wedge dz_n \ , \ \varphi = dy \wedge dz_1 \wedge \dots \wedge dz_n. \tag{10}$$

We also have a radial vector field Y on $K(M)$, given at a point $m \in K(M)$ by the vector m (viewed as a tangent vector to the linear fiber over $\pi(m)$). We have $i_Y\rho = 0$. Also the Lie derivative $\mathcal{L}_Y\rho = \rho$. So

$$\rho = i_Y d\rho = i_Y\varphi. \tag{11}$$

For M a Kähler-Einstein manifold with positive scalar curvature E. Calabi has constructed a complete Ricci-flat Kähler metric on $K(M)$ (see [4] and [14], p. 108). The metric is constructed as follows:

Definition 3.1.1. The connection on $K(M)$ induces a horizontal distribution for the projection π, with the corresponding splitting of the tangent bundle of $K(M)$ into horizontal and vertical distributions. We can identify the horizontal space at each point $m \in K(M)$ with the tangent space to M at $\pi(m)$. Let $r^2 : K(M) \to \mathbb{R}_+$ be the square of the length of an element in $K(M)$ and $u : \mathbb{R}_+ \to \mathbb{R}_+$ be a positive smooth function with a positive first derivative. We define the metric ω_u on $K(M)$ as follows: we put the horizontal and the vertical distributions to be orthogonal. On the horizontal distribution we define the metric to be $u(r^2)\pi^*(\omega)$ and on the vertical distribution we define it to be $2t^{-1}u'(r^2)\omega'$. Here ω is the Kähler-Einstein metric on M, t is the scalar curvature of ω, divided by $2n$ (see Equation (5)) and ω' is the induced metric on the linear fibers of π.

The Kähler-Einstein condition ensures that the corresponding 2-form ω_u defining this metric on $K(M)$ is closed, i.e., the metric is Kähler. If we take $u(r^2) = (tr^2 + l)^{\frac{1}{n+1}}$, $l > 0$ then for the corresponding metric ω_u on $K(M)$ we will have that the holomorphic $(n+1,0)$-form φ has constant length, hence φ is parallel and ω_u is Ricci flat. This metric is called the Calabi metric on $K(M)$.

From now on we endow $K(M)$ with a Kähler metric ω_u as above for any choice of the function u. We have a certain class of submanifolds of $K(M)$, called the Special Lagrangian (SLag) submanifolds of $K(M)$. We make the following general definition, taken from [8]:

Definition 3.1.2. Let (N^{2k}, ω) be a Kähler manifold of complex dimension k with a non-vanishing holomorphic $(k, 0)$-form φ. $(N^{2k}, \omega, \varphi)$ is called an almost Calabi-Yau manifold. Let P^k be an k-dimensional submanifold of N. Then P is called a Special Lagrangian (SLag) submanifold of N iff

$$\omega|_P = 0 \ , \ \mathrm{Im}\varphi|_P = 0.$$

We will not discuss the properties of SLag submanifolds here but refer the interested reader to [8] and [16].

We have an almost Calabi-Yau manifold $(K(M), \omega_u, \varphi)$ with the Kähler form ω_u and a holomorphic volume form φ constructed above. Let L be an oriented Lagrangian submanifold of M and let $\kappa_L : L \to K(M)$ be the canonical section (see Equation (7)). We define a submanifold $L^K \subset K(M)$ by

$$L^K = (m \in K(M) | m = a\kappa_L(l) \ \text{ for } \ l \in L \ , \ a \in \mathbb{R}). \tag{12}$$

We have the following:

Lemma 3.1.1 ([7]). *L is a minimal Lagrangian submanifold of M iff L^K is a Special Lagrangian submanifold of $K(M)$*

Proof. First we note that L^K is Special, i.e., $\mathrm{Im}\varphi|_{L^K} = 0$. Indeed $\mathrm{Im}\rho|_{L^K} = 0$ and hence $\mathrm{Im}\varphi|_{L^K} = d(\mathrm{Im}\rho|_{L^K}) = 0$.

We now prove that L^K is Lagrangian iff L is minimal. Let m be a point on L^K, $l = \pi(m) \in L$ and $m = a\kappa_L(l)$ for $a \in \mathbb{R}$. The tangent space to L^K at m is spanned by $\kappa_L(l)$ (viewed as a vertical vector in $T_m K(M)$) and by the vectors of the form $(e + a\nabla_e \kappa_L)$. Here e is any tangent vector to L at l (viewed as an element of the horizontal distribution of $T_m K(M)$) and $\nabla_e \kappa_L$ lives in the vertical distribution of $T_m K(M)$.

Let ξ be the connection 1-form on L defined by κ_L i.e., $\nabla\kappa_L = \xi \otimes \kappa_L$. Then ξ is a purely imaginary 1-form on L and $\nabla_e \kappa_l = \xi(e)\kappa_L$ for $e \in T_l L$. From this we easily deduce that L^K is a Lagrangian submanifold of $K(M)$ iff ξ vanishes, which by Lemma 2.1.2 is equivalent to L being a minimal submanifold of M. q.e.d.

The manifold L^K is invariant under the flow of the vector field Y on $K(M)$ (which is just scaling of $K(M)$ by real numbers). Vice versa we have the following:

Lemma 3.1.2. *Let L' be a Special Lagrangian submanifold of $K(M) - M$, invariant under the flow of Y. Then $L = \pi(L')$ is an (immersed) minimal Lagrangian submanifold of M.*

Proof. Let $m \in L'$. Since L' is Lagrangian and Y is in the tangent space $T_m L'$ then the tangent space to L' at m clearly decomposes as

$$T_m L' = \mathrm{span}(Y) \oplus T'$$

and T' is a subspace of the horizontal distribution to $K(M)$ at m. The space $\pi_*(T')$ can be viewed as the tangent space to L at $l = \pi(m)$. Clearly this tangent space $T_l L$ is Lagrangian, i.e., L is Lagrangian. Also L' was Special and Equation (11) tells us that $i_Y \varphi = \rho$. Thus m (viewed as an $(n,0)$-form on M at l) restricts to a real n-form on $T_l L$, i.e., $m \in L^K$. Hence locally L' coincides with L^K. From Lemma 3.1.1 we deduce that L is minimal. q.e.d.

3.2 Torus actions and the symplectic reduction

In the previous section we showed how to find minimal Lagrangian submanifolds of M from certain SLag submanifolds of $K(M)$. In this section we will see that if we have a structure-preserving torus action on M then we can find T-invariant SLag submanifolds of $K(M)$ from SLag submanifolds of a certain symplectic reduction of $K(M)$.

Let T^k act on M. Then this action induces a T^k-action on $K(M)$. Let $\mathcal{T}$ be the Lie algebra of T, $v \in \mathcal{T}$, X_v be its flow vector field on M and X'_v be its flow vector field on $K(M)$. Let $t > 0$ be the scalar curvature of M, divided by $2n$. Recall from Section 2, Lemma 2.1.1 that we have a natural moment map μ for the T^k-action on M given by

$$\mu \otimes v = -it^{-1}\mathrm{div}(X_v). \tag{13}$$

Here $v \in \mathcal{T}$ is an element of the Lie algebra of T, X_v if the flow vector field on M associated to v and $\otimes$ is the pairing between the Lie algebra and the dual Lie algebra of T. Let us now return to $K(M)$. The T^k-action on M induces a T^k-action on $K(M)$. We need to compute the moment map for the T^k-action on $(K(M), \omega_u)$ (see Definition 3.1.1). We have the following lemma, which we proved in [7]. Here we give a more direct proof, which we present for reader's convenience:

Lemma 3.2.1. *The map $\mu' = -it^{-1}u\pi^{-1}(\sigma) = u\pi^{-1}(\mu)$ is a moment map for the T^k-action on $(K(M), \omega_u)$.*

Proof. Let $v \in \mathcal{T}$ be an element in the Lie algebra of T, X_v be the associated flow vector field on M and let X'_v be the associated flow vector field on $K(M)$ for the T-action on $K(M)$. Let m be a point on $K(M)$.

It is enough to prove that for any tangent vector X in the tangent space $T_mK(M)$ to $K(M)$ at m we have $d\mu'(X)\otimes v = \omega_u(X'_v, X)$. Now the tangent space $T_mK(M)$ naturally decomposes into the direct sum $T_mK(M) = H \oplus V$ of a horizontal and a vertical distributions for the connection on $K(M)$ at m. By the construction of ω_u (Definition 3.1.1) we have that H and V are ω_u-orthogonal. We will prove that $d\mu'(X)\otimes v = \omega_u(X'_v, X)$ in two cases: X is in H or X is in V.

Case 1. $X \in H$. We have $\mu' = u(r^2)\pi^{-1}\mu$. Since $X \in H$ we have $X(r^2) = 0$. Hence

$$d\mu'(X)\otimes v = u(r^2)d\mu(\pi_* X)\otimes v = u(r^2)\omega(X_v, \pi_* X).$$

Also we have $\omega_u(X'_v, X) = u(r^2)\omega(X_v, \pi_* X)$ and we are done.

Case 2. $X \in V$. In that case we have $d\pi^{-1}(\mu)(X) = 0$ and $X(r^2) = 2\langle X, Y\rangle$. Here $Y \in V$ is the vector field as in (11) and $\langle\, ,\rangle$ is the Riemannian metric on V. So

$$\begin{aligned} d\mu'(X)\otimes v &= \langle X, Y\rangle 2u'(r^2)\mu\otimes v \\ &= -2iu'(r^2)t^{-1}\mathrm{div}(X_v)\omega'(Y, iX). \end{aligned} \tag{14}$$

Here ω' is the symplectic form on the linear fiber of $K(M)$ through m as before.

To compute $\omega_u(X'_v, X)$ we need to understand what is the vertical component of X'_v at m. We claim that this component is equal to $-\mathrm{div}(X_v)Y$. It is enough to prove this claim in case X_v doesn't vanish. Consider the trajectory m_t of m for the X'_v-flow on $K(M)$. We can view m_t as a section of $K(M)$ over the trajectory l_t of the X_v-flow on M through $l = \pi(m)$. Moreover we have $\mathcal{L}_{X_v} m_t = 0$. We also have that the vertical component of X'_v is equal to $\nabla_{X_v} m_t$. But since $\mathcal{L}_{X_v} m_t = 0$ we have by Proposition 2.1.2 of Section 2 that $\nabla_{X_v} m_t = -\mathrm{div}(X_v)m_t$ and the claim follows. So we have

$$\begin{aligned} \omega_u(X'_v, X) &= -2t^{-1}u'(r^2)\omega'(\mathrm{div}(X_v)Y, X) \\ &= -2u'(r^2)t^{-1}\omega'(i\mathrm{div}(X_v)Y, iX). \end{aligned} \tag{15}$$

Since $\mathrm{div}(X_v)$ is purely imaginary, the right hand side of Equation (15) is equal to $-2iu'(r^2)t^{-1}\mathrm{div}(X_v)\omega'(Y, iX)$. Comparing Equations (14) and (15) we get our claim. q.e.d.

Remark. We have seen in the proof of the previous lemma that if we pick an element v in the Lie Algebra of T^k then on $K(M) - M$ the vector field X'_v is horizontal precisely along the zero set of $\mu' \otimes v$. This fact will be useful later on.

Let now L' be a (connected) SLag submanifold of $K(M)-M$, invariant under the T-action and under the Y-flow. Since L' is Lagrangian and T-invariant, the moment map μ' is constant on L'. But $\mu' = u\pi^{-1}(\mu)$ and $Y(\mu') = 2r^2u'\pi^{-1}(\mu)$. So we must have $\pi^{-1}(\mu) = 0$ on L'. Thus L' must live on the zero set of the moment map μ'. Also from this argument and Lemma 3.1.1 we get that a T^k-invariant minimal Lagrangian submanifold L of M must live on the zero set of the moment map μ. Since those zero sets will be important in all subsequent discussions, let us assign them names:

Definition 3.2.1. Let $Z \subset M$ be the zero set of the moment map μ on M and let $Z' = \pi^{-1}(Z) \subset K(M)$ be the zero set of the moment map μ' on $K(M)$. Let $Z'' = Z' - Z \subset K(M)$. Let $Q = Z''/T$. Let X be the intersection of the unit circle bundle of $K(M)$ with Z' and let $S = X/T \subset Q$.

We will illustrate the relation between various sets we introduced in a commutative diagram:

$$\begin{array}{ccc} X \subset Z'' \subset Z' & \xrightarrow{\nu'} & S \subset Q \subset Z'/T \\ {\scriptstyle\pi}\Big\downarrow & & {\scriptstyle\pi'}\Big\downarrow \\ Z & \xrightarrow{\nu} & M_{\text{red}} = Z/T. \end{array} \tag{16}$$

From now on we assume that T acts freely on Z''. This implies that Z'' and Q are smooth. We will demonstrate examples where this holds in Section 4. We have a symplectic reduction $M_{\text{red}} = Z/T$ and a (smooth) symplectic reduction $Q = Z''/T$, endowed with a Kähler metric ω_u^{red} (see (16)). We will now see that Q has a natural non-vanishing holomorphic $(n+1-k,0)$-form φ', which a differential of a certain $(n,0)$-form ρ' on Q.

Let $v_1, ..., v_k$ be a basis for the Lie algebra $\mathcal{T}$ of T^k and let $X'_1, ..., X'_k$ be the corresponding flow vector fields on $K(M)$. Let $\varphi^* = i_{X'_1} \dots i_{X'_k}\varphi$ be an $(n-k+1,0)$-form on $K(M)$, obtained by contracting φ by $X'_1, \dots, X'_k$. Let $\rho^* = i_{X'_1} \dots i_{X'_k}\rho$. We claim that

$$\varphi^* = (-1)^k d\rho^*. \tag{17}$$

We prove this by induction on k. Namely let $\varphi^*_l = i_{X'_1} \dots i_{X'_l}\varphi$ and $\rho^*_l = i_{X'_1} \dots i_{X'_l}\rho$. We claim that $\varphi^*_l = (-1)^l d\rho^*_l$. For $l = 1$ we have that

ρ is X_1'-invariant. Hence

$$0 = \mathcal{L}_{X_1'}\rho = d\rho_1^* + \varphi_1^*.$$

Now we use induction. The form ρ_{l-1}^* is X_l'-invariant. Hence

$$0 = \mathcal{L}_{X_l'}\rho_{l-1}^* = (-1)^{l-1} d\rho_l^* + \varphi_l^*$$

and we are done by induction.

Both φ^* and ρ^* are T-invariant. Let $\nu : Z \to M_{\mathrm{red}}$ and $\nu' : Z'' \to Q$ be the quotient maps (see (16)). One easily sees that there is a unique $(n-k+1, 0)$-form φ' on Q and a unique $(n-k, 0)$-form ρ' on Q such that

$$(\nu')^*(\varphi') = \varphi^* \ , \ (\nu')^*(\rho') = \rho^* \ , \ \varphi' = (-1)^k d\rho'. \tag{18}$$

We state our observations as a Proposition:

Proposition 3.2.1. *$(Q, \omega_u^{\mathrm{red}}, \varphi')$ is an almost Calabi-Yau manifold as in Definition 3.1.2.*

The vector field Y is tangent to Z'' and T-invariant, hence it projects to a vector field Y' on Q. We had $i_Y\varphi = \rho$ on $K(M)$ (see Equation (11)). Hence we have

$$i_{Y'}\varphi' = (-1)^k \rho' \tag{19}$$

on Q. We obviously have the following:

Lemma 3.2.2. *Let L'' be a SLag submanifold of Q, invariant under the Y'-flow. Then $L' = (\nu')^{-1}(L'')$ is a SLag submanifold of $K(M)$, invariant under T^k and under the Y-flow. Here $\nu' : Z'' \to Q$ is the quotient map.*

3.3 Complexity one actions and periodic orbits

We continue to assume that we have a structure-preserving T^k-action on a compact Kähler-Einstein n-fold M with positive scalar curvature. In the previous section we have shown that one can reduce the problem of finding T^k and Y-invariant SLag submanifolds of $K(M)$ to finding Y'-invariant SLag submanifolds of Q (provided that T^k acts freely on Z'', see (16)). In this section we assume that $k = n-1$. We continue to assume that T^{n-1} acts freely on Z''. Let $X \subset Z''$ and $S = X/T \subset Q$

as in Definition 3.2.1. Thus S is a smooth compact 3-dimensional submanifold of a non-compact almost Calabi-Yau 2-fold Q. We will show that there is a vector field W on S such that there is a correspondence between Y'-invariant SLag submanifolds of Q and the trajectories of the W-flow on S.

As we saw the tangent bundle of $K(M)$ decomposes as a direct sum $V \oplus H$ of the vertical and the horizontal distributions. Let U be the image of the Lie algebra of T under the differential of the action on $K(M)$. At a point $z \in Z''$, $U(z)$ is an $(n-1)$-dimensional vector subspace of $T_z K(M)$, and it is contained in the horizontal distribution H at z (since on $\pi(Z'')$ the moment map μ vanishes, see Remark after the proof of Lemma 3.2.1). Also the Kähler form ω_u restricts to 0 on U. Let U^c be the complexification of U in the tangent bundle to $K(M)$. Then U^c can be viewed as a complex $(n-1)$-dimensional vector bundle over Z''. Let H' be the orthogonal complement of U^c in the horizontal distribution H along Z''. Then the tangent bundle of $K(M)$ along Z'' splits a direct sum $V \oplus H' \oplus U^c$. Also the quotient of $V \oplus H'$ under the T-action can be identified with the tangent bundle to the symplectic reduction $Q = Z''/T$. Since H' and V are T-invariant the tangent bundle to Q splits as a direct sum of 2 complex line bundles:

$$TQ = V \oplus H'. \tag{20}$$

Also V and H' are orthogonal both with respect to the symplectic form ω_u^{red} and the Riemannian metric on Q.

There is a natural circle action on X (see (16)), given by the multiplication by complex numbers of absolute value 1 on $K(M)$. This action is T-invariant, hence it induces a circle action on $S = X/T$. Let F be the vector field generating this action on S. Then $F = J(Y')$ (here J is the complex structure on Q and Y' is a vector field on Q as in Lemma 3.2.2). Also both Y' and F are in the vertical distribution V along S and the tangent bundle TS of S splits as a direct sum

$$TS = H' \oplus \mathrm{span}(F). \tag{21}$$

Here H' is the horizontal distribution of Q along S. Let γ be some path in S and let γ^Q be the orbit of γ under the Y'-flow in Q. We wish to understand when γ^Q is a SLag submanifold of Q. Let W be a tangent vector to γ. Clearly for γ^Q to be Lagrangian we need W to be ω_u^{red}-orthogonal to Y', hence W must live in the horizontal distribution H' along S. The form $\rho' = (-1)^{n-1} i_{Y'} \varphi'$ (see (19)) is a (non-zero)

$(1,0)$-form on H'. Hence the form $\mathrm{Im}\rho'$ has a 1-dimensional kernel in H'. Hence for γ^Q to be Special we need W to belong to the kernel of $\mathrm{Im}\rho'$. We can normalize W such that $\mathrm{Re}\rho'(W) = 1$. Those conditions give rise to a non-vanishing horizontal vector field W on S:

Definition 3.3.1. Let W be a vector field on S living in the horizontal distribution H' along S (see (21)) such that $\mathrm{Im}\rho'(W) = 0$ and $\mathrm{Re}\rho'(W) = 1$.

Let γ be a trajectory of W on S and consider $\gamma^Q \subset Q$ to be the image of γ under the Y'-flow. The forms ω_u^{red} and φ' vanish on γ^Q along $\gamma \subset \gamma^Q$. Also the Y'-flow preserves the horizontal distribution and $\mathcal{L}_{Y'}\rho' = \rho'$. From this we easily deduce that γ^Q is a Y'-invariant SLag submanifold of Q. From Lemmas 3.1.2 and 3.2.2 we get the following:

Lemma 3.3.1. *Let γ be a trajectory of W on S. Then $L_\gamma = \pi((\nu')^{-1}(\gamma))$ (see (16)) is an immersed minimal Lagrangian submanifold of M. If γ is periodic then L_γ is an immersed minimal Lagrangian torus.*

There is one general relation among trajectories of W, which will later be important: Consider the circle action on $K(M)$ as before. The $(n,0)$-form ρ is equivariant with respect to this action, i.e., if $\lambda \in S^1$ then $\lambda^*(\rho) = \lambda\rho$. So ρ' is also equivariant with respect to the circle action on Q. Also this action preserves the horizontal distribution H' on S (see Equation (21)). Consider an element $-1 \in S^1$. Then $-1^*(\rho') = -\rho'$. From this we deduce that the -1-action on S reverses the vector field W (see Definition 3.3.1), i.e., $-1_*(W) = -W$. We summarize this as:

Proposition 3.3.1. *The -1-action on S sends W-trajectories to W-trajectories, but it reverses their directions.*

4. Toric K-E manifolds

4.1 A generic subtorus

Let M be a Kähler-Einstein manifold with positive scalar curvature. In Section 3.3 we saw that if we have a T^{n-1}-action on M then one can construct minimal Lagrangian submanifolds of M from trajectories of the vector field W on S. In order to do this we needed T^{n-1} to act freely on Z'' (see Definition 3.2.1). In this section we assume that M is toric, i.e., we have an effective structure-preserving T^n-action on M.

For recent results on toric K-E manifolds we refer the reader to [17] and [3]. We will find a certain "generic" $(n-1)$-subtorus $T^{n-1} \subset T^n$ whose action on M accords with our requirements (in particular it will act freely on Z''). We will use this subtorus T^{n-1} to construct immersed, T^{n-1}-invariant minimal Lagrangian submanifolds. But first we note that there is a unique minimal Lagrangian torus, invariant under the whole of T^n.

Suppose L is a regular orbit of the T^n-action on M (i.e., an orbit with a finite stabilizer). Then the canonical moment map μ for the T^n-action on M (see (13)) is constant on L, hence L is a Lagrangian submanifold of M. Suppose that L is a minimal submanifold of M. As we have seen in Section 3.2 we must have $\mu = 0$ on L i.e., $L \subset \mu^{-1}(0)$. By Atyiah's result [1], $\mu^{-1}(0)$ is connected, hence $L = \mu^{-1}(0)$. So if a regular orbit, which is a minimal submanifold of M exists, it must coincide with $\mu^{-1}(0)$. We note that such an orbit does exist:

Lemma 4.1.1 ([7])**.** *There exists* (*a unique*) *regular orbit L of the T^n-action on M, which a is a minimal Lagrangian submanifold of M. L is precisely the zero set of the canonical moment map μ for the T^n-action on M as in* (13)*.*

Our next goal is to find a certain $(n-1)$-dimensional subtorus $T^{n-1} \subset T^n$ to proceed with the constructions in Section 3.3. This is done in the following lemma:

Lemma 4.1.2. *Let M^{2n} be a K-E manifold with an effective T^n-action as above. Then there is an $(n-1)$- torus $T^{n-1} \subset T^n$ such that*

i) *The differential of the T^{n-1}-action on M is injective along Z and T^n acts freely on Z''.*

ii) *There is an element v in the Lie algebra of T^{n-1} such that the flow vector field X_v doesn't have a constant length along Z.*

Here Z is the zero set of the canonical moment map for the T^{n-1} action on M as in (13) *and Z'' is given as in Definition 3.2.1.*

Remark. Condition ii) in the lemma will be later used to show that certain minimal Lagrangian tori we shall construct have Killing fields of non-constant length, hence they are not flat.

Proof. Let $T^{n-1} \subset T^n$ be some $(n-1)$-torus. First we prove that if the differential of the T^{n-1}-action on M is injective along Z (the zero set of the canonical moment map of T^{n-1}), then the T^n-action on Z''

is free. Suppose not. Then there is a point $l \in Z''$ and an element $1 \neq t \in T^n$ s.t. $t \cdot l = l$. In that case t also preserves the points on the T^{n-1}-orbit through l. The tangent space P to this orbit at l is in the horizontal distribution at l (since we are on the zero set of the moment map of T^{n-1}). Also $\omega_u|_P = 0$. So the differential dt of the t-action at l acts trivially on the complexification P^c of P in the tangent space $T_l K(M)$. Also dt acts trivially on the vertical distribution $V(l)$ at l. The vector space $P^c \oplus V(l)$ is a complex vector space of dimension n and dt acts trivially on it. Also dt preserves the holomorphic volume form φ on $K(M)$ at l. Hence dt is trivial at l. Hence t acts trivially on $K(M)$ and on M, but the T^n-action on M was effective- a contradiction.

Next we wish to understand for which $(n-1)$-tori $T^{n-1} \subset T^n$ the differential of the T^{n-1}-action is injective along the zero set Z of the canonical moment map of T^{n-1}. Let $\mathcal{T}^*$ be the dual Lie algebra of T^n and let $\Lambda \subset \mathcal{T}^*$ be the weight lattice of T^n. Any element $0 \neq v \in \Lambda$ defines an $(n-1)$-torus $T_v \subset T^n$ such that v vanishes on the Lie algebra of T_v. Let μ be the canonical moment map of T^n and μ_v be the canonical moment map of T_v. Then μ_v is just the restriction of μ to the dual Lie algebra of $T^{n-1} = T_v$. It is therefore clear that μ_v vanishes at a point $l \in M$ iff $\mu(l)$ is proportional to v. Since M is a toric variety, the moment polytope is convex and has no faces in the interior. Since 0 is in the interior of the moment polytope by Lemma 4.1.1, it is clear that $Z = \mu^{-1}[t_1 v, t_2 v]$ with $t_1 < 0 < t_2$. For any $t_1 < t < t_2$ the value tv is in the interior of the moment polytope, while $t_1 v$ and $t_2 v$ are not.

Let $0 \neq v \in \Lambda$ and suppose the line $\mathrm{span}(v) \subset \mathcal{T}^*$ doesn't intersect any of the $(n-2)$-faces of the moment polytope of μ. This means that any point in Z has either a trivial or a 1-dimensional stabilizer in T^n. We claim that in this case the differential of the $T^{n-1} = T_v$-action is injective along Z. Suppose not. Then there is a point $l \in Z$ and a vector $0 \neq w$ in the Lie algebra of T^{n-1} such that the flow vector field X_w vanishes at l. Since $l \in Z$ the flow vector field X'_w of w on $K(M)$ is horizontal along $\pi^{-1}(l) \subset K(M)$ (see Remark after the proof of Lemma 3.2.1). Hence X'_w vanishes along $\pi^{-1}(l)$. Let $g = \exp(tw)$ for some $t \in \mathbb{R}$. Then the g-action on $\pi^{-1}(l)$ is trivial. But this means that the differential dg of the g-action on the tangent space $T_l M$ has Jacobian 1. Also g acts trivially on the orbit L' of the T^n-action through l. The tangent space $T_l L'$ of L' at l is $(n-1)$-dimensional and ω restricts to 0 on it. Hence its complexification $(T_l L')^c$ in $T_l M$ is a complex $(n-1)$-dimensional space and dg acts trivially on it. Also dg has Jacobian 1. Hence dg is trivial, hence g acts trivially- a contradiction.

A generic line in the projective space $P\mathcal{T}^*$ doesn't intersect the $(n-2)$-faces of the moment polytope of μ. Also the set of lines passing through points of the weight lattice Λ is dense in $P\mathcal{T}^*$. So we can find $v \in \Lambda$ so that Condition i) in our lemma holds for $T^{n-1} = T_v$. In order to ensure that ii) holds, consider a point b in the $(n-2)$-face of the moment polytope (here we use the fact that $n \geq 2$). The T^n-orbit $\mu^{-1}(b)$ has a stabilizer of dimension at least 2. Hence we can find a vector $0 \neq w \in \mathcal{T}$ in the Lie algebra of T^n such that $b \otimes w = 0$ and the flow vector field X_w vanishes along $\mu^{-1}(b)$. We can find a sequence of elements $v_k \in \Lambda$ such that the lines $(v_k) = \mathrm{span}(v_k)$ do not intersect the $(n-2)$-faces of the moment polytope and (v_k) converge to the line $(b) = \mathrm{span}(b)$ in $P\mathcal{T}^*$. We can also find a sequence of vectors $w_k \in \mathcal{T}$ such that $v_k \otimes w_k = 0$ and w_k converge to w.

Each v_k defines an $(n-1)$-torus $T_k \subset T$. By our construction the vectors w_k are in the Lie algebra of T_k. Let μ_k be the canonical moment map of T_k, and Z_k be the zero set of μ_k. We can find points n_k on Z_k such that n_k converge to a point $n \in \mu^{-1}(b)$. Let X_k be the flow vector field of w_k. Then the length of X_k at points n_k goes to 0 as $k \to \infty$. On the other hand the torus $L = \mu^{-1}(0)$ coming from Lemma 4.1.1 is contained in all of Z_k. Moreover the lengths of X_k along L are a-priori bounded from below. So we deduce that for k large enough the torus $T^{n-1} = T_k$ satisfies Conditions i) and ii) of the lemma. q.e.d.

4.2 Flow-invariant function and its level sets

We continue to assume that we have a toric Kähler-Einstein n-fold M. From now on we pick a sub-torus $T^{n-1} = T_v \subset T$ satisfying the conditions of Lemma 4.1.2. Here v is a non-zero element in the dual Lie algebra of T^n and v vanishes on the Lie algebra of T^{n-1}. To avoid confusion we explicitly note that the subsets X, Z, Z'' and their reductions S, M_{red}, Q we've introduced in Equation (16) are for the T^{n-1}-action on M and $K(M)$. From Lemma 3.3.1 we deduce that one can construct minimal Lagrangian submanifolds of M from the trajectories of the vector field W on S. Moreover periodic orbits of W give rise to immersed minimal Lagrangian tori in M. Our goal therefor is to understand the periodic orbits of W on the compact 3-manifold S. In this section we will show that there is a function f on S which is constant on the orbits of the W-flow on S. We will also investigate the salient features of the level sets of f. We construct the function f as follows:

The circle $R = T/T^{n-1}$ acts freely on Q and on S by Lemma 4.1.2.

Definition 4.2.1. Let $w \neq 0$ be some non-zero element in the Lie algebra of R. Let A_w be the flow vector field for the w-action on Q and on S.

The vector fields A_w and W commute (since the A_w-flow is structure-preserving and in particular in preserves the defining conditions 3.3.1 for W). We also have a $(1,0)$-form ρ' and a holomorphic $(2,0)$-form φ' on Q with $\varphi' = (-1)^{(n-1)} d\rho'$ (see Equation (18)). The flow of A_w preserves ρ' and φ'. A key point in finding periodic trajectories of W is the fact that there is a function on S constant along the trajectories of W:

Lemma 4.2.1. *Let $h = \rho'(A_w)$ and $f = \mathrm{Re}(h)$. Then h is an S^1-equivariant function on S and f is constant along the trajectories of W on S.*

Proof. The fact that h is S^1-equivariant on S follows from the fact that A_w is S^1-invariant (since the S^1 and the T^n-actions on $K(M)$ commute) and ρ' is an S^1-equivariant $(1,0)$-form on Q (see the discussion in the end of Section 3.3).

Next we prove that f is constant along the trajectories of W on S. We have:

$$0 = \mathcal{L}_{A_w}\rho' = d(i_{A_w}\rho') + i_{A_w} d\rho' = dh + (-1)^{(n-1)} i_{A_w}\varphi'$$

(see Equation (18)). So $dh = (-1)^n i_{A_w}\varphi'$. So $dh(W) = (-1)^n \varphi'(A_w, W)$. The vector field A_w is in the tangent bundle to S, hence we can decompose it into $A_w = A_w^H + \lambda F$. Here A_w^H is the horizontal part of A_w (i.e., the part in the distribution H'; see Equation (21)), F is the generator of the S^1-action on S and $\lambda \in \mathbb{R}$. W is horizontal and H' is a 1-dimensional complex vector bundle. The form φ' is a $(2,0)$-form on Q. Hence $\varphi'(A_w^H, W) = 0$. Also $F = JY'$. Hence $\varphi'(F, W) = i\varphi'(Y', W)$. By the construction of W we had that $\varphi'(Y', W)$ is real. From all this we deduce that $dh(W)$ is purely imaginary, hence $df(W) = 0$. q.e.d.

From the previous lemma we deduce that the trajectories of W live on level sets of the function f. Next we need to understand those level sets in more detail.

We had our symplectic reductions $M_{\mathrm{red}} = Z/T^{n-1}$ and Q and we have a natural projection $\pi' : Q \to M_{\mathrm{red}}$ (see (16)). Let v be an element of the weight lattice Λ of $\mathcal{T}^*$ defining the torus $T^{n-1} = T_v$ (so v vanishes on the Lie algebra of T^{n-1}). As we have seen in the proof of

Lemma 4.1.2, Z is equal to $\mu^{-1}[t_1v, t_2v]$ for $t_1 < 0 < t_2$. Here μ is the canonical moment map for the T^n-action on M. We have the following:

Proposition-Definition 4.2.1. *Let $Z_0 = \mu^{-1}(t_1v, t_2v)$ and $Z'_0 = \pi^{-1}(Z_0) \subset K(M)$. T^{n-1} acts freely on Z_0 and we have $M_0 = Z_0/T^{n-1} \subset M_{\mathrm{red}}$, which is the smooth part of M_{red}. Let $S_0 = (\pi')^{-1}(M_0) \bigcap S = (Z'_0 \bigcap X)/T^{n-1}$ (see (16)). Then the circle S^1 acts freely on S_0 and S_0 is an S^1 fiber bundle over M_0.*

Let $\overline{K_i} = \pi^{-1}(\mu^{-1}(t_iv)) \bigcap X$, $i = 1, 2$ (here μ is the canonical moment map for the T^n-action on M). Let $a_i = \mu^{-1}(t_iv)/T^{n-1} \in M_{\mathrm{red}}$ and $K_i = \overline{K_i}/T^{n-1} = (\pi')^{-1}(a_i) \bigcap S$ (see (16)). Then each a_i is a point in M_{red}, K_i is a circle in S and S^1 acts locally freely on K_i.

Proof. We have that T^n acts freely on M_0, and so in particular T^{n-1} acts freely on M_0 and so S^1 acts freely on S_0, i.e., S_0 is an S^1 fiber bundle over M_0.

Also $\mu^{-1}(t_iv)$ is connected (being a T^n orbit on M) and is $(n-1)$-dimensional and T^{n-1} acts locally freely on it (see Lemma 4.1.2). From this it is clear that $a_i = \mu^{-1}(t_iv)/T^{n-1} \in M_{\mathrm{red}}$ is 1 point. Also the T^{n-1} action is free on $\overline{K_i}$ and $\overline{K_i}$ is n-dimensional and connected, hence K_i is a circle in S. Also since the T^{n-1} action on $\mu^{-1}(t_iv)$ is locally free we deduce that the generating vector field F of the S^1-action on $\overline{K_i}$ (see (21)) is not contained in the image of the differential of the T^{n-1}-action on $\overline{K_i}$, hence the S^1-action on $K_i = \overline{K_i}/T^{n-1}$ is locally free. q.e.d.

We will represent the setup in the previous Proposition-Definition in a commutative diagram.

$$\begin{array}{ccc} Z'' \bigcap X = (\overline{K_i}) \bigcup (Z'_0 \bigcap X) & \xrightarrow[\nu']{} & S = (K_i) \bigcup S_0 \\ \Big\downarrow \pi & & \Big\downarrow \pi' \\ Z = (\mu^{-1}(t_iv)) \bigcup Z_0 & \xrightarrow[\nu]{} & M_{\mathrm{red}} = (a_i) \bigcup M_0. \end{array} \tag{22}$$

On Z_0 (see Proposition-Definition 4.2.1) we have an oriented Lagrangian distribution D, given by the image of the Lie algebra $\mathcal{T}$ of T^n under the differential of the T^n-action on M. This distribution gives rise to a section κ of $K(M)$ over Z_0 such that κ restricts to the Riemannian volume form on this Lagrangian distribution. This section is T^n-invariant and has constant length $\sqrt{2}^n$. Hence it gives rise to an $R = T^n/T^{n-1}$-

invariant section

$$\kappa' = \kappa/\sqrt{2}^n : M_0 \to S_0. \tag{23}$$

By definition κ' restricts to a positive real n-form on the Lagrangian distribution D. From this we deduce that $h = \rho'(A_w)$ is real and positive along $\kappa'(M_0)$.

We can normalize w (the generator of the Lie algebra of $R = T^n/T^{n-1}$) such that $v \otimes w = 1$ (here v as before is the element of the dual Lie algebra of T^n defining the sub-torus T^{n-1}). On M_{red} we have a function

$$\tau = \mu \otimes w. \tag{24}$$

The image of τ is the interval $[t_1, t_2]$. For each $t_1 \leq t \leq t_2$ the level set $\tau^{-1}(t)$ is an orbit of the $R = T^n/T^{n-1}$-action on M_{red}. We define:

Definition 4.2.2. Let $L' = \tau^{-1}(0)$, $L_+ = \kappa'(L') \subset S$ (see Equation (23)) and $L_- = (-1) \cdot L_+$.

Each $L_\pm$ is an orbit of the R-action on S. Also at points of $L_\pm$ the vector field A_w is horizontal (since $\mu \otimes w = 0$ on L') and $\rho'(A_w)$ is real. The vector field W also satisfies those properties (see Definition 3.3.1), hence W is proportional to A_w along $L_\pm$. So we see that $L_\pm$ are trajectories of W (of course the minimal Lagrangian torus of M coming from these trajectories as in Lemma 3.3.1 is the T^n-invariant minimal Lagrangian torus $L = \mu^{-1}(0)$). We have the following:

Lemma 4.2.2. *The differential df of f is non-vanishing on $S - (L_- \bigcup L_+)$.*

Proof. We have seen in the proof of Lemma 4.2.1 that

$$dh = (-1)^n i_{A_w} \varphi'.$$

On $S' = S \bigcap (\pi')^{-1}(M_{\text{red}} - L')$ the vertical part of the vector field A_w doesn't vanish (see the Remark after the proof of Lemma 3.2.1). Hence the form $i_{A_w}\varphi'$ restricts as a non-vanishing $(1,0)$-form on the horizontal distribution H' along S' (see Equation (21)). From this it is clear that $df|_{H'} \neq 0$ on S'.

On $S \bigcap (\pi')^{-1}(L') - (L_- \bigcup L_+)$ the function h is not real. Also h is equivariant with respect to the S^1-action. Let F be the vector field generating the S^1-action as before. Then the derivative of $f = \text{Re} h$ is non-zero in the direction of F. q.e.d.

f attains a constant value f_+ along L_+ and a value $f_- = -f_+$ along $L_- = (-1) \cdot L_+$. Since S is compact and connected, it is clear from Lemma 4.2.2 that f_+ is the absolute maximum of f, attained only at L_+, and f_- is the absolute minimum of f, attained only at L_-. Also for any $s \in (f_-, f_+)$, the level set $\Sigma_s = f^{-1}(s)$ is smooth. We will also need the following fact:

Proposition 4.2.1. $f|_{K_i} = 0$ (*see Proposition-Definition* 4.2.1).

Proof. Proposition-Definition 4.2.1 tells that the circles K_i are orbits for the S^1-action on S, hence the tangent space to K_i lives in the vertical distribution along S (see (21)). Also the $R = T^n/T^{n-1}$-action preserves K_i, so the vector field A_w is tangent to K_i, so A_w is vertical along K_i. But from this we deduce that $h = \rho'(A_w) = 0$ at K_i, and so $f = 0$ at K_i. q.e.d.

4.3 Periodic trajectories and the main theorem

In the previous section we saw that the orbits of the vector field W on S live on the level sets of the function f on S. Moreover all these level sets except $L_\pm = f^{-1}(f_\pm)$ are smooth 2-dimensional submanifolds of S. We make the following definition:

Definition 4.3.1. Let $\Phi = f^{-1}(f_-, f_+)$.

Take any point $m \in \Phi$ and consider the level set Σ_s of f passing through m. From Lemma 4.2.2 we know that Σ_s is smooth. Let Σ_s^0 be the connected component of Σ_s containing m. The vector field W is tangent to Σ_s^0. We have a free $R = T^n/T^{n-1}$-action on Σ_s^0, and this action preserves the vector field W.

Proposition 4.3.1. *The vector field A_w (see Definition 4.2.1) is transversal to W at all points of Σ_s^0.*

Proof. Let $m' \in \Sigma_s^0$. If $m' \in S \bigcap (\pi')^{-1}(M_{\text{red}} - L')$ (see Definition 4.2.2), then the vector field A_w is not horizontal at m', so it can't be proportional to W. If $m' \in S \bigcap (\pi')^{-1}(L') - (L_- \bigcup L_+)$ then $h = \rho'(A_w)$ is not real, while $\rho'(W)$ is real. So again A_w and W can't be proportional. q.e.d.

From Proposition 4.3.1 we get that the quotient of Σ_s^0 by the R-action is a circle and W projects to a non-vanishing vector field on it. From this we deduce that the W-trajectory on S starting at m will intersect the R-orbit of m. Suppose it intersects this orbit for the first time at a point $\xi(m)m$, $\xi(m) \in R$. This gives rise to a well-defined function

$$\xi : \Phi \to R = T^n/T^{n-1}. \tag{25}$$

Clearly ξ is continuous, R-invariant and constant along the trajectories of W. Also we have seen in Proposition 3.3.1 that the -1-action on S sends W-trajectories to W-trajectories in the reverse direction. From this we easily deduce that for any point $m \in \Phi$:

$$\xi(-1 \cdot m) = \xi(m)^{-1}. \tag{26}$$

Obviously the trajectory through m is periodic iff $\xi(m)$ is a root of unity in R. Let R' be the set of roots of unity in R. Since ξ is continuous, $\xi^{-1}(R')$ will be everywhere dense in Φ unless ξ assumes a constant value not in R' on some open subset of Φ. The next lemma shows that it is impossible:

Lemma 4.3.1. *Suppose that ξ is constant on some open set $U \subset \Phi$. Then ξ is equal to a constant g on the whole of Φ and $g^2 = 1$.*

Proof. Let $S_+ = f^{-1}(0, f_+)$, $S_- = f^{-1}(f_-, 0)$. Thus $S_- = -1 \cdot S_+$. Suppose that ξ is constant on some open set $U \in \Phi$. Then ξ is constant on some open ball U' either in S_+ or in S_-. We can assume w.l.o.g. that $U' \subset S_+$. We note that S_+ is connected. In fact S_+ is given by

$$S_+ = (\kappa'(a)e^{i\theta} | a \in M_0\ ,\ \ -\pi/2 < \theta < \pi/2) - L_+.$$

Here κ' is given by Equation (23) and M_0 is the smooth part of the symplectic reduction M_{red} (see (22)). First we prove that ξ is a constant g on S_+. Let A_w^H be the horizontal part of the vector field A_w (see Definition 4.2.1). Since $S_+ \subset \pi^{-1}(M_0)$ we deduce that A_w^H doesn't vanish on S_+. We also note that on S_+ the vector field A_w^H cannot be proportional to JW. Indeed suppose that $A_w^H = \lambda JW$ for some $\lambda \in \mathbb{R}$ at some point $m \in S_+$. Then

$$h(m) = \rho'(A_w) = \rho'(A_w^H) = i\lambda\rho'(W).$$

So $h(m)$ is purely imaginary, hence $f(m) = 0$ - a contradiction. Since both A_w^H and W lie in H' (see Equation (21)), which is a complex 1-dimensional distribution, we deduce that we can find a function $b : S_+ \to \mathbb{R}$ such that on S_+ the vector field W is pointwise proportional to a vector field

$$W' = A_w^H + bJA_w^H. \tag{27}$$

Hence the trajectories of W' and W coincide. We also note that the W'-trajectories live on level sets of f which are compact submanifolds of S_+. Hence the W'-flow is complete on S_+. We will use W' instead of W to prove that ξ is constant on S_+.

Definition 4.3.2. For a point $m \in S_+$ let $t(m)$ be the time for the W'-flow to hit the R-orbit of m for the first time.

We have the following:

Lemma 4.3.2. *For any point $m \in S_+$ we have*

$$\xi(m) = \exp(t(m)w).$$

Proof. Let γ be the trajectory of W' through m and $\gamma' = \pi'(\gamma)$ be the corresponding path in M_0 (here $\pi' : S_+ \to M_0$ is the natural projection; see (22)). We have a free R-action on M_0 and the corresponding flow vector field B_w for the w-flow on M_0 (here w is the generator of the Lie algebra of R). We obviously have $\pi'_*(A^H_w) = B_w$. Hence the tangent field to γ' is $B_w + bJB_w$.

The R-action on M_0 is Hamiltonian with the moment map $\tau = \mu \otimes w$ (see (24)). Also the B_w-flow on M_0 commutes with the complex structure J on M_0. Hence the vector fields B_w and JB_w commute. Let $P_1 = \gamma'(0)$ and $P_2 = \gamma'(t(m))$. Then $P_2 = \xi(m)P_1$. We will prove that $P_2 = \exp(t(m)w)P_1$ and since the R-action on M_0 is free this would prove that $\exp(t(m)w) = \xi(m)$.

Let $\exp(xJB_w)$ be the time x flow of JB_w. Note that the JB_w-flow is not complete. In fact we have

$$JB_w(\tau) = \omega_{\text{red}}(B_w, JB_w) = |B_w|^2 > 0.$$

So τ increases on the JB_w-trajectories. Let

$$c(r) = \int_{[0,r]} b(\gamma(t))dt \ for\ 0 \le r \le t(m). \tag{28}$$

Consider a path

$$\gamma''(r) = \exp(c(r)(JB_w))\exp(rw)(P_1)$$

(note that we flow P_1 with respect to rB_w first). Then γ'' is a path on M_0, $\gamma''(r)$ is defined for small values of r, $\gamma''(0) = P_1$ and the tangent vector to γ'' is $B_w + b(\gamma(r))JB_w$. So γ'' coincides with γ' whenever it is defined.

Next we prove that $\gamma''(r)$ is defined for $r \leq t(m)$. Suppose on γ' the function τ (see Equation (24)) ranges between the values s_1 and s_2. Then $t_1 < s_1$ and $s_2 < t_2$ (see Proposition-Definition 4.2.1). Pick any r for which $\gamma''(r)$ is defined and consider the path $\exp(tJB_w)\exp(rw)(P_1)$ for t ranging between 0 and $c(r)$. The function τ is increasing along the path, and on the endpoints its values are between s_1 and s_2. Hence this path lives in the compact set $A = \tau^{-1}[s_1, s_2]$ in M_0. From this one can easily deduce that $\gamma''(r)$ is well defined for all $0 \leq r \leq t(m)$ and coincides with $\gamma'(r)$. In particular $P_2 = \exp(c(t(m))JB_w)\exp(t(m)w)(P_1)$. Now $P_2 = \xi(m)P_1$, hence

$$\tau(P_2) = \tau(P_1) = \tau(\exp(t(m)w)P_1)$$

and τ increases on the trajectories of JB_w. So we get that $c(t(m)) = 0$, i.e., $P_2 = \exp(t(m)w)P_1$. q.e.d.

Now we can prove that ξ is constant on S_+. Since ξ is constant on U' we get from Lemma 4.3.2 that $t(m)$ (see Definition 4.3.2) is a constant t on U'. Let ϕ_t be the time t flow of W' on S_+. Consider the map $\chi = \exp(-tw)\cdot\phi_t : S_+ \to S_+$. S_+ is a connected real analytic manifold and χ is a real analytic map. Also χ is the identity map on U'. So we deduce that χ is the identity map. So ϕ_t is the multiplication by $g = \exp(tw)$ on S_+. From this we easily deduce that $\xi = g$ on S_+.

So ξ assumes a constant value g on S_+, hence by (26) it assumes a constant value g^{-1} on $S_- = -1\cdot S_+$. Let $\Delta = f^{-1}(0)$. Then Δ is the common boundary of S_+ and S_- in Φ. Since ξ is continuous, we have $g = g^{-1}$, i.e., $g^2 = 1$. q.e.d.

From this we get an immediate corollary

Corollary 4.3.1. *The set $\xi^{-1}(R')$ is everywhere dense in Φ. Here R' is the set of roots of unity in R.*

We are now ready to state and prove our main result:

Theorem 4.3.1. *Let M^{2n} $(n \geq 2)$ be a K-E manifold with positive scalar curvature with an effective structure preserving T^n-action. Then precisely one regular orbit L of the T^n-action is a minimal Lagrangian submanifold of M. Moreover there is an $(n-1)$-torus $T^{n-1} \subset T^n$ and a sequence of non-flat T^{n-1}-invariant immersed minimal Lagrangian tori $L_k \subset M$ such that L_k locally converge to L (in particular the supremum of sectional curvatures of L_k and the distance between L and L_k goes to 0 as $k \to \infty$).*

Proof. Choose a torus T^{n-1} which satisfies the conditions of Lemma 4.1.2. By Corollary 4.3.1 we can choose a sequence of points $m_k \in \xi^{-1}(R')$ such that m_k converge to a point m in L_+(see Definition 4.2.2). The W-trajectories γ_k through m_k are periodic and live on level sets Σ_k of f with Σ_k converging to L_+ in the distance topology. From this we easily see that γ_k locally converge to the trajectory L_+. One deduces that the immersed minimal Lagrangian tori L_k which γ_k define as in Lemma 3.3.1 locally converge to the minimal, T-invariant orbit L.

Finally we prove that L_k are not flat. From Lemma 4.1.2 we get a vector v in the Lie algebra of T^{n-1} such that the flow vector field X_v of v doesn't have a constant length on Z. The vector field X_v along L_k is a Killing vector field of L_k. To prove that L_k is not flat it is enough to prove that $|X_v|^2$ is non-constant on L_k.

The function $|X_v|^2$ is T^n-invariant on Z. Thus it projects to an R-invariant function on M_{red}, i.e., it can be viewed as a function of $\tau = \mu \otimes w$ (see Equation (24)) on M_{red}. Also $|X_v|^2$ is a real analytic function on $M_0 = \tau^{-1}(t_1, t_2)$. Since $|X_v|^2$ is non-constant, it is nowhere a locally constant function of τ. We finally claim that the function τ is non-constant on the projection $\pi'(\gamma_k)$ of γ_k onto M_{red}. Indeed if τ were constant on $\pi'(\gamma_k)$ that means that $\pi'(\gamma_k)$ is contained in the R-orbit β on M_{red}. But the vector field W (the tangent vector filed to γ_k) is horizontal and non-vanishing on S. Hence if $\pi'(\gamma_k)$ is contained in the R-orbit β on M_{red} that means that $\pi'(\gamma_k)$ coincides with β. But that by Lemma 3.3.1 means that the preimage $\nu^{-1}(\beta)$ of β in M is a minimal Lagrangian orbit of the T^n-action on M and this orbit is different from L since γ_k are different from $L_\pm$. But $L = \mu^{-1}(0)$ is the unique minimal Lagrangian T^n-orbit in M- a contradiction. Thus $|X_v|^2$ is non-constant on L_k and we are done. q.e.d.

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